JOURNAL OF APPROXIMATION THEORY 40, 155–157 (1984)

The Limit of Approximation with Weights Large on Nodes

CHARLES B. DUNHAM

Computer Science Department, The University of Western Ontario, London, Ontario N6A 5B7, Canada

Communicated by T. J. Rivlin

Received March 12, 1982

Let X be a compact topological space and $C(X)$ the space of real (complex) continuous functions on X. For $g \in C(X)$ define

$$\|g\| = \sup\{|g(x)|: x \in X\}. \tag{1}$$

Let F be an approximating function with parameter space P (a non-empty closed subset of (real or complex) n-space) such that $F(A, \cdot) \in C(X)$ for $A \in P$. Let Z be a finite subset of X. The problem of approximation with interpolation on Z is: given $f \in C(X)$, find a parameter A^* minimizing $e(A) = \|f - F(A, \cdot)\|$ subject to the constraint

$$F(A, z) = f(z), \qquad z \in Z. \tag{*}$$

Such a parameter A^* is called best and $F(A^*, \cdot)$ is called a best approximation to f with interpolation on Z [1, 50ff.].

We will only consider approximation by approximating families (F, P) satisfying Young's condition [2; 6, p. 27]. These include finite-dimensional linear families, real families unisolvent on an interval $[\alpha, \beta]$, and tame rationals [3, 4]. Let $\| \ \|_c$ be the maximum norm on n-space.

DEFINITION. (F, P) satisfies *Young's condition* if

(i) $\|A^k\|_c \to \infty$ implies $\|F(A^k, \cdot)\| \to \infty$,

(ii) $A \in P$, $A^k \in P$, $\{A^k\} \to A$ implies $F(A^k, \cdot)$ converges uniformly to $F(A, \cdot)$ on X.

By doing weighted uniform approximation with weights large on Z, we can produce approximations $F(A, \cdot)$ with $f(z) - F(A, z)$ small. This suggests that a limit of weighted approximation with weights $\to\infty$ on Z would be best to f with respect to (*).

0021-9045/84 $3.00

Best approximations with respect to weight w, which is $\geqslant 1$ on Z and 1 off Z, exist by standard arguments.

THEOREM. *Let (F, P) satisfy Young's condition.*

Let $F(B, \cdot)$ interpolate f on Z. Let $\{w_k\}$ be a sequence of positive weight functions on X such that $w_k(x) = 1$ for $x \notin Z$ and $w_k(z) \to \infty$ for $z \in Z$. Let A^k be best with respect to w_k. Then $\{A^k\}$ has an accumulation point and if A^0 is an accumulation point, A^0 is best and there is a sequence $\{k(j)\}$ such that $\{F(A^{k(j)}, \cdot)\} \to F(A^0, \cdot)$ uniformly on X.

Proof. Suppose $\{A^k\}$ is unbounded. By arguments of [2], $\|f - F(A^k, \cdot)\|$ is unbounded, hence $\|w_k(f - F(A^k, \cdot)\|$ is unbounded. But

$$\|w_k(f - F(B, \cdot))\| = \|f - F(B, \cdot)\| \tag{2}$$

and this would contradict A^k being best with respect to w_k.

As $\{A^k\}$ is bounded, it has an accumulation point A^0. By taking a subsequence if necessary, we can assume $\{A^k\} \to A^0$. We claim A^0 satisfies (*). Suppose not then there is $\varepsilon > 0$ and $z \in Z$ such that $|f(z) - F(A^0, z)| > \varepsilon$. Hence

$$w_k[f(z) - F(A^k, z)] \to \infty$$

and this (with (2)) contradicts optimality of A^k. Next suppose A^0 is not best with respect to (*). Then there is C satisfying (*) and $\varepsilon > 0$ such that

$$\|f - F(C, \cdot)\| < \|f - F(A^0, \cdot)\| - \varepsilon. \tag{3}$$

Now we claim

$$\liminf_{k \to \infty} \|w_k(f - F(A^k, \cdot))\| \geqslant \|f - F(A^0, \cdot)\|. \tag{4}$$

This is proven by letting x be a point at which $\|f - F(A^0, \cdot)\|$ is attained, then

$$w_k(x)(f(x) - F(A^k, x)) \to f(x) - F(A^0, x).$$

Now (3, 4) contradict optimality of A^k. A^0 is best and uniform convergence follows by definition.

L. Keener [5] has announced results for a special case of our problem.

If X is finite, we can use programs for weighted discrete uniform approximation to get best interpolating approximations.

The conclusion of the theorem may not hold if Young's condition fails.

EXAMPLE. Let $X = \{0, 1/2, 1\}$, $Z = \{0\}$, and $f(x) = x^2$. Let $F(A, x) =$

$a_1 + a_2 x$, $P = \{(a_1, a_2)\colon\ a_1 \neq 0\} \cup \{(0, 0)\}$. Let $w_k(x) = k$ if $x = 0$ and $w_k(x) = 1$ for $x \neq 0$. There is a best approximation p_k with respect to weight w_k by first-degree polynomials and $w_k(f - p_k)$ must alternate twice on X, hence $p_k \in (F, P)$. By drawing a diagram it is seen that p_k is not near the zero function. But the only interpolating approximation by (F, P) is zero.

References

1. B. L. Chalmers and G. D. Taylor, Uniform approximation with constraints, *Jahresber. Deutsch. Math.-Verein* **81** (1979), 49–86.
2. C. B. Dunham, The limit of non-linear Chebyshev approximation on subsets, *Aequationes Math.* **9** (1973), 60–63.
3. C. B. Dunham, Chebyshev approximation by rationals with constrained denominators, *J. Approx. Theory* **37** (1983), 5–11.
4. E. H. Kaufman, Jr. and G. D. Taylor, Uniform approximation by rational functions having restricted denominators, *J. Approx. Theory* **32** (1981), 9–26.
5. L. Keener, The best interpolating approximation is a limit of best weighted approximations, *Canad. Math. Bull.* **25** (1982), 502–503.
6. J. Rice, "The Approximation of Functions," Vol. 1, Addison–Wesley, Reading, Mass., 1964.